APPENDIX A

Average of Wilson loop integrals

In this appendix we show that

$$\langle\langle \exp(-i\oint_c A_\mu \mathrm{d}x_\mu\rangle\rangle = \int \mathcal{D}A \exp(-i\oint_c A_\mu \mathrm{d}x_\mu)\exp(-\frac{i\theta}{4\pi^2}\int \mathrm{d}^3x\, C.S.) = (-1)^{2s}e^{is\Omega} \ , \tag{A.1}$$

where $\theta = \pi/2s$. The above integral over A is a Gaussian one and formally integrating, we get

$$\exp\left(\frac{i\pi^2}{\theta}\int \mathrm{d}x_\mu \int \mathrm{d}y_\nu [d^{-1}]_{\mu\nu}\right) \ . \tag{A.2}$$

But since "d" is a singular operator it is defined only if we fix a gauge. For that we choose the gauge $\partial_\mu A_\mu = 0$. In this gauge d could be inverted to write

$$[d^{-1}]_{\mu\nu} = \frac{1}{4\pi}\epsilon_{\mu\nu\lambda}\frac{(x-y)_\lambda}{|x-y|^3} \ , \tag{A.3}$$

so that

$$\langle\langle exp(-i\oint_c A_\mu \mathrm{d}x_\mu\rangle\rangle = exp\left(i\frac{\pi^2}{\theta}W\right) \tag{A.4}$$

where

$$W = \frac{1}{4\pi}\oint \mathrm{dx}_\mu \oint \mathrm{dy}_\nu \epsilon_{\mu\nu\lambda}\frac{(x-y)_\lambda}{|\mathbf{x}-\mathbf{y}|^3} \ , \tag{A.5}$$

where both line integrals are done over the same curve. By definition this is known as the "writhing number" of the curve. We would like to connect this to another quantity known as the "twist". For this we will first prove a relation for "ribbons", namely

$$Lk = W + T \ , \tag{A.6}$$

where Lk stands for "linking number" and T for "twist". This relation was first proved by White[38], but we give a simplified proof following an interesting paper by Frank-Kamenetskii and Vologodoskii[39].

Let $r(t)$ represent the curve under question. We will assume that the curve is sufficiently smooth so that $r(t)$ can be differentiated as many times as is necessary. The parametrisation is such that it represents a closed curve of parametric length l. That is

$$r(0) = r(l) \ , r'(0) = r'(l) \ , \text{ and } \ r''(0) = r''(l) \tag{A.7}$$

where 'prime' stands for differentiation with respect to the parameter. We can choose the parametrisation such that the speed $|r'(t)| = 1$.

For this let $\mathbf{n}(t)$ be a unit vector normal to the curve $r(t)$ so that $y(t) = r(t)+\epsilon n(t)$, where ϵ is a small constant, represent a second closed curve. That is we have

$$r(l) + \epsilon n(l) = r(0) + \epsilon n(0) \ , \mathbf{r}(t) \cdot \mathbf{n}(t) = 0 \text{ and} \mathbf{n} \cdot \mathbf{n} = 1 \tag{A.8}$$

We assume that the acceleration $|\mathbf{r}''(t)| \leq c$, where c is a constant and ϵ to be so small that the type of linkage does not change as $\epsilon \to 0$.

Now we can think of a ribbon with $x(t)$ and $y(t)$ as the boundaries. Then we can calculate the Gauss integral in the limiting case of $\epsilon \to 0$. What we have to evaluate is

$$
\begin{aligned}
& \frac{1}{4\pi} \oint \mathrm{d}x_\mu \oint \mathrm{d}y_\nu \epsilon_{\mu\nu\lambda} \frac{(x-y)_\lambda}{|\mathbf{x}-\mathbf{y}|^3} \\
= & \frac{1}{4\pi} \int \mathrm{d}t \int \mathrm{d}s \epsilon_{\mu\nu\lambda} \frac{r'_\mu(t)(r'_\nu(s) + \epsilon n'_\nu(s))(r_\lambda(t) - r_\lambda(s) - \epsilon n_\lambda(s))}{|\mathbf{r}(t) - \mathbf{r}(s) - \epsilon \mathbf{n}(s)|} \qquad \text{(A.9)}
\end{aligned}
$$

By definition this is equal to the linking number of the two curves forming the edges of the ribbon. Let us evaluate the integral in another way also. For that we will divide the region of integration with respect to t into two regions, the first being a small neighbourhood of s,

$$s - \delta \le t \le s + \delta \ ,$$

and the second being part of the integral where the integrand has got no singularities, provided the curve does not intersect itself.

In the neighbourhood of s we can have the Taylor expansion,

$$\mathbf{r}(t) = \mathbf{r}(s) + \mathbf{r}'(s)(t-s)$$

$$\mathbf{r}'(t) = \mathbf{r}'(s)$$

We also assume that $\mathbf{r}'(s)$ is in the direction of $\mathbf{r}(s)$ itself and so we can write

$$\mathbf{r}'(s) \cdot \mathbf{n}(s) = 0$$

since $\mathbf{n}(s)$ is taken to be normal to the curve $\mathbf{r}(s)$

So in the integral we can write

$$\mathbf{r}(t) - \mathbf{r}(s) - \epsilon\mathbf{n}(s) = \mathbf{r}'(s)(t-s) - \epsilon\mathbf{n}(\mathbf{s})$$

and

$$|\mathbf{r}(t)-\mathbf{r}(s)-\epsilon\mathbf{n}(s)|^3 = \{|\mathbf{r}'(s)|^2(t-s)^2 - 2\epsilon(t-s)\mathbf{r}'(s)\cdot\mathbf{n}(s) + \epsilon^2\mathbf{n}\cdot\mathbf{n}\}^{\frac{3}{2}}$$
$$= \left((t-s)^2 - \epsilon^2\right)^{\frac{3}{2}} \qquad \text{(A.10)}$$

Using these, the integral in the δ neighbourhood could be written as,

$$-\frac{1}{4\pi}\epsilon^2 \int \mathrm{ds}\, \epsilon_{\mu\nu\lambda} r'_\mu(s) n'_\nu(s) n_\lambda(s) \int_{s-\delta}^{s+\delta} \frac{\mathrm{dt}}{((\mathrm{t}-\mathrm{s})^2+\epsilon^2)^{\frac{3}{2}}} \qquad \text{(A.11)}$$

Now we can do the t integral assuming $\epsilon \ll \delta$

$$\int_{s-\delta}^{s+\delta} \frac{\mathrm{dt}}{((\mathrm{t}-\mathrm{s})^2+\epsilon^2)^{\frac{3}{2}}} = \int_{-\delta}^{+\delta} \frac{\mathrm{d}\tau}{(\tau^2+\epsilon^2)^{\frac{3}{2}}} = \frac{2}{\epsilon^2} \qquad \text{(A.12)}$$

so that the δ neighbourhood integral is,

$$\frac{1}{2\pi}\int \mathrm{ds}\epsilon_{\mu\nu\lambda} r'_\mu(s) n_\nu(s) n'_\lambda(s) \qquad \text{(A.13)}$$

The integrand in the remaining part of the parameter space is non-singular and the integral could be written as, (by taking the limit $\epsilon \to 0$)

$$\frac{1}{4\pi}\int_0^l \mathrm{dt} \int_0^l \mathrm{ds}\, \epsilon_{\mu\nu\lambda} \frac{r'_\mu(t) r'_\nu(s)(r_\lambda(t)-r_\lambda(s))}{|\mathbf{r}(t)-\mathbf{r}(s)|^3} \qquad \text{(A.14)}$$

This integral is over the parameter ranges excluding the δ neighbourhood. But since this integral is well defined as $\delta \to 0$ we can extend this to the whole region of the parameter space. Thus we have,

$$Lk = \frac{1}{2\pi}\int \mathrm{ds}\, \epsilon_{\mu\nu\lambda} r'_\mu(s) n_\nu(s) n'_\lambda(s)$$
$$\frac{1}{4\pi}\int_0^l \mathrm{dt} \int_0^l \mathrm{ds}\, \epsilon_{\mu\nu\lambda} \frac{r'_\mu(t) r'_\nu(s)(r_\lambda(t)-r_\lambda(s))}{|\mathbf{r}(t)-\mathbf{r}(s)|^3} \qquad \text{(A.15)}$$

The second term by definition is the "writhe" W and the first term is what is known as the "torsion", T So we have,

$$W = Lk - T \tag{A.16}$$

Note that both Lk and T depends on the way we construct the ribbon associated to the curve, but W is independent of that.

Now we would like to understand the "torsion" in more detail. We have,

$$T = \frac{1}{2\pi}\int_0^l \mathrm{d}t\epsilon_{\mu\nu\lambda}r'_\mu(s)n_\nu(s)n'_\lambda(s) \tag{A.17}$$

Now let us define

$$\frac{\mathrm{d}r_\mu}{\mathrm{d}t} = u_\mu \tag{A.18}$$

Now let us introduce the unitary matrices U and $U^\dagger$ such that,

$$\mathbf{u} = U^\dagger\sigma^3 U = u_\mu\sigma^\mu \quad \text{and} \quad \mathbf{n} = U^\dagger\sigma^2 U = n_\mu\sigma^\mu \tag{A.19}$$

where σ^μ are Pauli matrices. Note that the conditions $u_\mu u_\mu = 1$, $n_\mu n_\mu = 1$ and $n_\mu u_\mu = 0$ are automatically satisfied. Using the relations,

$$[\sigma^\mu, \sigma^\nu] = 2i\epsilon_{\mu\nu\lambda}\sigma^\lambda \quad \text{and} \quad \mathrm{tr}(\sigma^\mu\sigma^\nu) = 2\delta_{\mu\nu} \tag{A.20}$$

we can write,

$$T = -\frac{i}{8\pi}\int \mathrm{d}t\,\mathrm{tr}\left(\mathbf{u}[\mathbf{n}, \mathbf{n}']\right) \tag{A.21}$$

With our definitions we have,

$$\mathrm{tr}\left(\mathbf{u}[\mathbf{n}, \mathbf{n}']\right) = 4\mathrm{tr}\left(\sigma^3 U' U^\dagger\right) = -8iL_t^3 \tag{A.22}$$

where $L_t = i\partial_t U U^\dagger$ and $L_t^\mu = \frac{1}{2}\mathrm{tr}(\sigma^\mu i\partial_t U U^\dagger)$ Now we can write,

$$T = -\frac{1}{\pi}\int_0^l \mathrm{dt}\, L_t^3 \tag{A.23}$$

Now if we introduce another variable s such that (t, s) spans the surface for which our curve is the boundary, and use Stoke's theorem we get,

$$T = -\frac{1}{\pi}\iint \mathrm{dtds}\,\left(\partial_t L_s^3 - \partial_s L_t^3\right) \tag{A.24}$$

But we have the relation,

$$\partial_t L_s^3 - \partial_s L_t^3 = -\frac{i}{2}\mathrm{tr}\left(\sigma^3[L_t, L_s]\right)$$

So we have

$$T = \frac{1}{\pi}\iint \mathrm{dtds}\,\frac{i}{2}\mathrm{tr}\left(\sigma^3[L_t, L_s]\right) \tag{A.25}$$

We have

$$\partial_t \mathbf{u} = \partial_t\left(U^\dagger\sigma^3 U\right) = iU^\dagger[L_t, \sigma^3]U \tag{A.26}$$

so that we can write,

$$\begin{aligned}
\mathrm{tr}\left(\mathbf{u}[\partial_t\mathbf{u}, \partial_s\mathbf{u}]\right) &= -\mathrm{tr}\left(U^\dagger\sigma^3 U\,\left[U^\dagger[L_t, \sigma^3]U,\, U^\dagger[L_s, \sigma^3]U\right]\right) \\
&= -\mathrm{tr}\left(\sigma^3\,\left[[L_t, \sigma^3],\, [L_s, \sigma^3]\right]\right) \\
&= 4\,\mathrm{tr}\left(\sigma^3[L_t, L_s]\right)
\end{aligned} \tag{A.27}$$

Substituting this in the expression for T, we get,

$$T = \frac{i}{8\pi}\iint \mathrm{dtds}\,\mathrm{tr}\left(\mathbf{u}[\partial_t\mathbf{u}, \partial_s\mathbf{u}]\right) \tag{A.28}$$

But we have

$$\mathrm{tr}\,(\mathbf{u}[\partial_t \mathbf{u}, \partial_s \mathbf{u}]) = 4i\, \epsilon_{\mu\nu\lambda}\, u_\mu\, \partial_t u_\nu\, \partial_s u_\lambda \tag{A.29}$$

So that we can write

$$T = -\frac{1}{2\pi}\int \mathrm{d}s\mathrm{d}t\, \epsilon_{\mu\nu\lambda} u_\mu \partial_t u_\nu \partial_s u_\lambda \tag{A.30}$$

Here the integrand in the RHS is the area element on a sphere so that we can write

$$T = -\frac{\Omega}{2\pi}$$

where Ω is equal to the solid angle for a unit sphere. So we get $W = m + \frac{\Omega}{2\pi}$ where m is an integer. We have

$$e^{\frac{i\pi^2}{\theta}} = e^{i2\pi s m} e^{is\Omega} \;, \tag{A.31}$$

for special values of θ viz. $\theta = \frac{\pi}{2s}$. Now we have to fix the value of the integer. Let us take the special case of the closed curve being a circle. In this case we have the writhe, $W = 0$. and $\Omega = 2\pi$. From the relation connecting W to Ω it follows that the integer is odd. For other curves which can be obtained from a circle by smooth deformations this integer will be the same. For other closed curves which can not be obtained from circle by smooth deformations this integer differs by an even integer. This is because deformation of this curve into a simple circle will involve crossing of the segment of the curve with another segment of the same curve and this process changes W be an even integer. In all the cases, the integer connecting W and Ω is an odd number. So we get the result that

$$\langle\langle exp\left(\oint_c A_\mu \mathrm{d}x_\mu\right)\rangle\rangle = e^{\frac{i\pi^2}{\theta}W} = (-1)^{2s} e^{is\Omega} \;. \tag{A.32}$$